OKA'S FIRST COHERENCE THEOREM

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CONTENTS

ABSTRACT

> I can illustrate the second approach with the same image of a nut to be opened. The first analogy which came to my mind is of immersing the nut in some softening liquid, and why not simply water? From time to time you rub so the liquid penetrates better, and otherwise you let time pass. The shell becomes more flexible through weeks and months – when the time is ripe, hand pressure is enough, the shell opens like a perfectly ripened avocado!
>
> A different image came to me a few weeks ago. The unknown thing to be known appeared to me as some stretch of earth or hard marl, resisting penetration... the sea advances insensibly in silence, nothing seems to happen, nothing moves, the water is so far off you hardly hear it... yet it finally surrounds the resistant substance.
>
> (Alexander Grothendieck, Récoltes et semailles, 1985–1987, pp. 552-3-1 The Rising Sea)

In that spirit, our goal is to soften the nut of sheaf theoretical methods in complex analysis, that is, we want to derive Oka's First Coherence Theorem so that at the very end all we got is a perfectly ripened avocado, ready to be spread upon our bread for breakfast. We try to be as detailed as possible in our presentation. The content of this document basically amounts to a summary of what is written in more sophisticated papers/books such as [1], [2] and [3]. With that being said, we will follow the presentation of [1] quite closely. However, at times it was tried to give more details so as to make the contained information more accessible to someone who has never seen sheaf theory before. The main struggle with the theory at hand is mostly one of digesting the abundance of definitions that will be thrown at the reader quite violently at times. However, we hope that the sea advances insensibly in silence and then, all of a sudden, it will have already surrounded the resistant lack of understanding, ready to wash it away.

1 LOCAL WEIERSTRASS THEORY

Watson. Come at once if convenient. If inconvenient, come all the same.

(Sherlock Holmes - Arthur Conan Doyle, The Adventure of the Creeping Man)

1.1 *Weierstrass Division Theorem*

We first introduce the notion of germs of holomorphic functions in several variables. This is rather simple and analogous to the one-variable case. We denote by $\mathcal{O}_0 = \mathbb{C}\{z_1, ..., z_d\}$ the $\mathbb{C}$-algebra of convergent power series around $0 \in \mathbb{C}^d$. In other words,

$$\mathbb{C}\{z_1, ..., z_d\} := \left\{ f = \sum_{\mu \in \mathbb{N}^d} a_\mu z^\mu \middle| f \text{ converges on some polydisc centered around } 0 \right\} \tag{1}$$

where $a_\mu \in \mathbb{C}$ and z^μ is a shorthand notation for $\prod z_j^{\mu_j}$. Furthermore, we will make use of the convention $\mathcal{O}'_0 = \mathbb{C}\{z_1, ..., z_{d-1}\}$ as we will give some special importance to the variable z_d in this section. We will usually write w instead of z_d. We may then consider both $\mathcal{O}'_0$ and the polynomial ring $\mathcal{O}'_0[w]$ as sub-rings of $\mathcal{O}_0$. For $r \in \mathcal{O}'_0[w]$ the degree of r in the variable w is denoted by $\deg(r)$. Note that if $f \in \mathcal{O}_0$ is such that $f(0) \neq 0$, then f is a unit in $\mathcal{O}_0$. In particular, whenever $g = g_0 + ... + g_b w^b \in \mathcal{O}'_0[w]$ such that $g_b(0) \neq 0$ we know that the leading coefficient g_b of the polynomial $g \in \mathcal{O}'_0[w]$ is a unit in $\mathcal{O}'_0$ and therefore the euclidean algorithm holds for g, that is, to every polynomial $f \in \mathcal{O}'_0[w]$ there are uniquely determined $q, r \in \mathcal{O}'_0[w]$ such that $f = qg + r$ with $\deg(r) < b$. The goal is to generalize this division theorem to convergent power series in w. In order to do so we must first introduce the notion of an order.

Definition 1. We say that a germ $f \in \mathcal{O}_0$ has order $b \in \mathbb{N}$ in the variable w, if

$$f = \sum_{\nu=0}^{\infty} f_\nu w^\nu \qquad f_0(0) = ... = f_{b-1}(0) = 0 \qquad f_b(0) \neq 0$$

Now we can state the famous:

Theorem 1 (Weierstrass Division Theorem). *If $g \in \mathcal{O}_0$ has order b in w, then for every germ $f \in \mathcal{O}_0$ there exist a uniquely determined germ $q \in \mathcal{O}_0$ and a polynomial $r \in \mathcal{O}'_0[w]$ such that*

$$f = qg + r \qquad deg(r) < b$$

In order to prove this Theorem some preparation is needed. First, let

$$f = \sum a_{\mu_1 ... \mu_d} z_1^{\mu_1} ... z_{d-1}^{\mu_{d-1}} w^{\mu_d} \tag{2}$$

be a formal power series with complex coefficients and let $\rho := (r_1, ..., r_d)$ be a d-tuple of positive real numbers. We use the notation:

$$\|f\|_\rho := \sum |a_{\mu_1 ... \mu_d}| r_1^{\mu_1} ... r_{d-1}^{\mu_{d-1}} r^{\mu_d} \qquad B_\rho := \{f \colon \|f\|_\rho < \infty\} \tag{3}$$

One can then prove, quite analogously to how we proved the single variable case in the lecture, that:

Lemma 1. *The space B_ρ is a Banach-algebra.*

Now each element $f \in B_p$ can be written as a power series in w (as we have already noted earlier):

$$f = \sum_0^\infty f_\nu w^\nu \qquad \|f\|_\rho = \sum_0^\infty \|f_\nu\|_{\rho'} r_d^\nu \text{ with } \rho' := (r_1, ..., r_{d-1}) \tag{4}$$

Using this notation we define for a given integer $b \geq 0$:

$$\widehat{f} := \sum_0^{b-1} f_\nu w^\nu \qquad \tilde{f} := \sum_b^\infty f_\nu w^{\nu - b} \tag{5}$$

Thus $\widehat{f}$ is a polynomial in w of degree $< b$ and by construction $f = \widehat{f} + w^b\tilde{f}$. Moreover, we always have

$$\|\widehat{f}\|_\rho \le \|f\|_\rho \qquad \|\tilde{f}\|_\rho \le r_d^{-b}\|f\|_\rho \tag{6}$$

That was all the preparation we needed:

Proof of Theorem 1. Let $0 < \varepsilon < 1$ be fixed. We first choose ρ such that $g \in B_\rho$. Then, by using the notation from before, we must have that $\tilde{g}$ is a unit in $\mathscr{O}_0$ (since $\tilde{g}(0) \ne 0$). We may assume without loss of generality that $\tilde{g}^{-1} \in B_p$ and we can furthermore arrange that

$$\|w^b - g\tilde{g}^{-1}\|_\rho \le r_d^b\varepsilon. \tag{7}$$

Now let $f \in \mathscr{O}_0$. Yet again we may assume that $f \in B_\rho$. We now define elements $v_j \in B_\rho$ recursively as follows:

$$v_0 := f \qquad v_{j+1} := (w^b - g\tilde{g}^{-1})\tilde{v}_j = -\widehat{g}\tilde{g}^{-1}\tilde{v}_j \tag{8}$$

Since we have $\|\tilde{v}_j\|_\rho \le r_d^{-b}\|v_j\|_\rho$ we conclude that $\|v_{j+1}\|_\rho \le \varepsilon\|v_j\|_\rho$. Thus, since B_ρ is a Banach space, $v := \sum_0^\infty v_j \in B_\rho$ exists. We then note that

$$v_j - v_{j+1} = (\widehat{v}_j + w^b\tilde{v}_j) - (-\widehat{g}\tilde{g}^{-1}\tilde{v}_j) = \widehat{v}_j + (w^b + \widehat{g}\tilde{g}^{-1})\tilde{v}_j = \widehat{v}_j + g\tilde{g}^{-1}\tilde{v}_j \tag{9}$$

Putting $q := \tilde{g}^{-1}\tilde{v} \in B_\rho, r := \widehat{v} \in B_\rho$, we immediately obtain:

$$f = \sum_0^\infty (v_j - v_{j+1}) = \sum_0^\infty (g\tilde{g}^{-1}\tilde{v}_j + \widehat{v}_j) = qg + r \tag{10}$$

where r is a polynomial of degree $< b$. It remains to prove uniqueness of q and r. For this it suffices to show that $qg + r = 0$ with $q \in \mathscr{O}_0, r \in \mathscr{O}'_0[w]$ and $\deg(r) < b$ implies $q = r = 0$. Again we may assume $q, g, r \in B_\rho$ for some suitable ρ. Since $g_b(0) \ne 0$ we may arrange $g_b^{-1} \in B_\rho$ and write $g = g_b(w^b + h)$ with $h \in B_\rho$ and $h(0) = 0$. Again we may choose ρ such that $\|h\|_\rho \le r_d^b\varepsilon$. We observe that $qg_bw^b + r = -qg_bh$ and $\deg(r) < b$ and therefore we infer:

$$M := \|qg_b\|_\rho r_d^b = \|qg_bw^b\|_\rho \le \|qg_bw^b + r\|_\rho = \|qg_bh\|_\rho \le \|qg_b\| r_d^b\varepsilon = M\varepsilon \tag{11}$$

As ε was strictly between 0 and 1 we must have that $M = 0$, that is, $qg_b = 0$. As g_b is a unit we conclude that $q = 0$. □

This theorem has tremendous impact and gives rise to a chain of nice consequences.

Corollary 1. *If $g \in \mathscr{O}_0$ is a germ of order b in w, then the Weierstrass Division Theorem gives rise to an $\mathscr{O}'_0$-module epimorphism $\mathscr{O}_0 \twoheadrightarrow \mathscr{O}'^b_0$ with kernel $\mathscr{O}_0 g$. This map will be referred to as the Weierstrass map.*

Proof. For $f \in \mathscr{O}_0$ we know by Theorem 1 that there is a uniquely determined germ $q \in \mathscr{O}_0$ and a uniquely determined polynomial $r \in \mathscr{O}'_0[w]$ so that $f = qg + r$ and $\deg(r) < b$. Now the polynomial r can be written as

$$r = \sum_{\nu=0}^{b-1} r_\nu w^\nu$$

for coefficients $r_\nu \in \mathscr{O}'_0$. The map

$$\mathscr{O}_0 \to \mathscr{O}'^b_0 \qquad f \mapsto (r_0, ..., r_{b-1})$$

is the desired epimorphism. □

One of the most important consequences of the Weierstrass Division Theorem is the (also very famous) Weierstrass Preparation Theorem. Luckily this second theorem follows quite easily from the first one.

Definition 2. A Weierstrass polynomial ω in w of degree $b \ge 1$ over $\mathscr{O}'_0$ is a polynomial of the form

$$\omega = w^b + a_1w^{b-1} + ... + a_b \in \mathscr{O}'_0[w] \qquad a_1(0) = ... = a_b(0) = 0$$

Lemma 2. *Suppose $\omega \in \mathscr{O}'_0[w]$ is a Weierstrass polynomial and $q \in \mathscr{O}_0$ is a germ such that $q\omega \in \mathscr{O}'_0[w]$, then $q \in \mathscr{O}'_0[w]$.*

Proof. Setting $f := q\omega$ we clearly see that $q\omega$ is the Weierstrass decomposition of f with respect to $\omega \in \mathscr{O}_0$. However, by assumption $f \in \mathscr{O}'_0[w]$ and thus by the euclidean algorithm we also have $f = \tilde{q}\omega + r$ with $\tilde{q}, r \in \mathscr{O}'_0[w]$ and therefore by uniqueness $q = \tilde{q}$. □

Theorem 2 (Weierstrass Preparation Theorem). *If $g \in \mathscr{O}_0$ is a germ of order $b \geq 1$ in w, then there exists a uniquely determined Weierstrass polynomial $\omega \in \mathscr{O}'_0[w]$ of degree b and a unit $e \in \mathscr{O}_0$ such that $g = e\omega$. In particular, if $g \in \mathscr{O}'_0[w]$ then $e \in \mathscr{O}'_0[w]$.*

Proof. By the Weierstrass division theorem we may write the monomial w^b as

$$w^b = qg + r \qquad \deg(r) < b$$

However, this immediately yields $w^b = q(0,w)g(0,w) + r(0,w)$. From that we may infer $r(0,w) = 0$ and $q(0,0) \neq 0$. In particular, this means that q is a unit in $\mathscr{O}_0$. Now if we define $e := 1/q \in \mathscr{O}_0$ and $\omega := w^b - r$, then $g = e\omega$. Lastly, if $g \in \mathscr{O}'_0[w]$ then $e \in \mathscr{O}'_0[w]$ by lemma 2. □

Just like the Weierstrass Division Theorem induced a map, so does the Weierstrass Preparation Theorem:

Corollary 2. *Let $g \in \mathscr{O}_0$ be a germ of order $b \geq 1$ and write $g = e\omega$ (by means of Theorem 2). There is a $\mathbb{C}$-ring isomorphism $\mathscr{O}'_0[w]/\mathscr{O}'_0[w]\omega \xrightarrow{\simeq} \mathscr{O}_0/\mathscr{O}_0 g$. Moreover, the Weierstrass polynomial ω is prime in $\mathscr{O}'_0[w]$ if and only if it is prime in $\mathscr{O}_0$.*

Proof. Note that the residue classes $f + \mathscr{O}_0 g \in \mathscr{O}_0/\mathscr{O}_0 g$ boil down to $f + \mathscr{O}_0 g = f + \mathscr{O}_0\omega$ since $e \in \mathscr{O}_0$ is a unit. Moreover, for $f \in \mathscr{O}_0$ we have by the Division Theorem 1 that $f = qg + r$ for $q \in \mathscr{O}_0$ and $r \in \mathscr{O}'_0[w]$ with $\deg(r) < b$. Using this we immediately see

$$\mathscr{O}'_0[w]/\mathscr{O}'_0[w]\omega = \{f + \mathscr{O}'_0[w]\omega \mid f \in \mathscr{O}'_0[w],\ \deg(f) < b\} \tag{12}$$

$$\simeq \{f + \mathscr{O}_0\omega \mid f \in \mathscr{O}'_0[w] \subset \mathscr{O}_0,\ \deg(f) < b\} = \mathscr{O}_0/\mathscr{O}_0 g \tag{13}$$

Now recall that an ideal P of some ring R is prime if and only if the quotient ring R/P is an integral domain. Since we have the isomorphism $\mathscr{O}'_0[w]/\mathscr{O}'_0[w]\omega \xrightarrow{\simeq} \mathscr{O}_0/\mathscr{O}_0\omega$, we know that $\mathscr{O}'_0[w]/\mathscr{O}'_0[w]\omega$ is an integral domain if and only if $\mathscr{O}_0/\mathscr{O}_0\omega$ is an integral domain. □

From the preceding results one may also verify that:

Corollary 3. *The ring $\mathscr{O}_0$ is both noetherian and factorial.*

Our next goal is to further refine the Division Theorem 1, since this will be vital for some of the later stages. In order to do so we will first need a Lemma:

Lemma 3 (Hensel's Lemma). *Suppose $\omega = w^b + a_1 w^{b-1} + ... + a_b \in \mathscr{O}'_0[w]$. Next write $\omega(0,w) = \prod (w - c_j)^{b_j}$ for distinct roots $c_1, ..., c_t \in \mathbb{C}$ and $b_1, ..., b_t \in \mathbb{N}$. Then there exist unique monic polynomials $\omega_1, ..., \omega_t \in \mathscr{O}'_0[w]$ of degree $b_1, ..., b_t$ such that*

$$\omega = \prod \omega_j \qquad \omega_l(0,w) = (w - c_l)^{b_l} \text{ for all } 1 \leq l \leq t$$

Proof. We will only sketch the proof. We verify the statement by induction on t. For $t = 1$ the statement trivially holds, so assume $t > 1$. Applying the Preparation Theorem to $\omega \in \mathscr{O}'_0[w - c_1]$ we get $\omega = \omega_1 e$ for $\omega_1, e \in \mathscr{O}'_0[w - c_1]$ with $\deg(\omega_1) = b_1$ (where ω_1 is a Weierstrass polynomial and e is a unit). Now one can deduce that e is a monic polynomial in w of degree $b_2 + ... + b_t$ with $e(0,w) = \prod_{j=2}^{t} (w - c_j)^{b_j}$. By induction hypothesis $e = \prod_{j=2}^{t} \omega_j$ with monic polynomials $\omega_j \in \mathscr{O}'_0[w]$ of degree b_j such that $\omega_j(0,w) = (w - c_j)^{b_j}$. □

Before stating the generalization of Theorem 1 we must agree on some notational conventions. Let $\omega \in \mathscr{O}_0'[w]$ be a monic polynomial of degree $b \geq 1$ and let $c_1,...,c_t \in \mathbb{C}$ denote the distinct roots of the polynomial $\omega(0,w) \in \mathbb{C}[w]$. Set $x_j := (0,c_j) \in \mathbb{C}^{d+1}$ and denote by $\mathscr{O}_{x_j}$ the ring of germs of holomorphic functions at x_j. If p is a polynomial $\mathscr{O}_0'[w]$, then let $p_{x_j} \in \mathscr{O}_{x_j}$ denote the induced germ at each point x_j for $1 \leq j \leq t$.

Theorem 3 (Generalized Division Theorem). *Let $\omega \in \mathscr{O}_0'[w]$ and $c_j \in \mathbb{C}$ be as above. Given arbitrary germs $f_j \in \mathscr{O}_{x_j}$ there exist uniquely determined germs $q_j \in \mathscr{O}_{x_j}$ and a uniquely determined polynomial $r \in \mathscr{O}_0[w]$ with deg$(r) < b$ such that for all $1 \leq j \leq t$ we have*

$$f_j = q_j\omega_{x_j} + r_{x_j} \tag{14}$$

Proof. By Hensel's Lemma 3 we know that there are monic polynomials $\omega_1,...,\omega_t \in \mathscr{O}_0[w]$ so that for all $1 \leq j \leq t$ we have

$$\omega_{x_j} = \prod_s \omega_{sx_j} \qquad \omega_j(0,w) = (w-c_j)^{b_j}$$

We first prove existence of the decomposition (14): Put $e_i := \prod_{j\neq i}\omega_j$. Of course, since $e_i(x_j) \neq 0$ we have by Theorem 1 that

$$f_j e_{jx_j}^{-1} = q_j'\omega_{jx_j} + r_{jx_j}$$

with $q_j' \in \mathscr{O}_{x_j}, r_j \in \mathscr{O}_0[w-c_j]$ and $\deg(r_j) < b_j$. Now put $e_{ij} := \prod\limits_{s\neq i,j} \omega_s$ and note that

$$f_j = q_j'\omega_{jx_j}e_{jx_j} + r_{jx_j}e_{jx_j} = \left(q_j' - \sum_{i\neq j} r_{ix_j}e_{ijx_j}\right) + \sum r_{ix_j}e_{ix_j} \tag{15}$$

Since by construction $q := q_j' - \sum\limits_{i\neq j} r_{ix_j}e_{ijx_j} \in \mathscr{O}_{x_j}$ and $r := \sum r_i e_i \in \mathscr{O}_0[w]$ and $\deg(r) < b$, the existence statement of (14) follows. Now we verify uniqueness: So suppose $0 = q_j\omega_{x_j} + r_{x_j}$ with $q_j \in \mathscr{O}_{x_j}$ and $r \in \mathscr{O}_0[w]$ with $\deg(r) < b$. Suppose $r \neq 0$, then $p_j := r/(\omega_1...\omega_j) \neq 0$ and for $2 \leq j \leq t$ we obtain

$$r = p_1\omega_1 \qquad p_{j-1} = p_j\omega_j \qquad r = p_t\omega \tag{16}$$

As we thus we have $r_{x_t} = -q_t\omega_{x_t} = p_{tx_t}\omega_{xt}$ and therefore by uniqueness of the Weierstrass decomposition $p_t = -q_t$ and in particular since $r_{x_j} = -q_j\omega_{x_j} = p_{jx_j}(\omega_1...\omega_j)_{x_j}$ we have, again by uniqueness, $p_{jx_j} = -q_j(\omega_{j+1}...\omega_t)_{x_j}$ for every $j < t$. However, note that $\omega_{jx_j} \in \mathscr{O}_0[w-c_j]$ is a Weierstrass polynomial and therefore by Lemma 2 (applied iteratively) combined with equation (16) we obtain $p_1,...,p_t \in \mathscr{O}_0[w]$. Thus $r = p_t\omega \in \mathscr{O}_0[w]\omega$, which yields uniqueness. □

We may phrase the preceding Theorem in even more abstract and elegant terms:

Corollary 4. *Let $\omega \in \mathscr{O}_0'[w]$ be a monic polynomial as given in the Generalized Division Theorem 3. Define the maps*

$$\pi\colon \mathscr{O}_0[w] \longrightarrow \bigoplus_{j=1}^{t} \mathscr{O}_{x_j}/\mathscr{O}_{x_j}\omega_{x_j} \qquad p \mapsto \sum\left(p_{x_j} + \mathscr{O}_{x_j}\omega_{x_j}\right) \tag{17}$$

$$\psi\colon \mathscr{O}_0^b \longrightarrow \bigoplus_{j=1}^{t} \mathscr{O}_{x_j}/\mathscr{O}_{x_j}\omega_{x_j} \qquad (r_0,...,r_{b-1}) \mapsto \pi\Big(\sum_{\nu=0}^{b-1} r_\nu w^\nu\Big) \tag{18}$$

Then ψ is an $\mathscr{O}_0$-module isomorphism and π gives rise to a $\mathbb{C}$-algebra isomorphism $\mathscr{O}_0[w]/\mathscr{O}_0[w]\omega \xrightarrow{\sim} \bigoplus\limits_{j=1}^{t} \mathscr{O}_{x_j}/\mathscr{O}_{x_j}\omega_{x_j}$.

2 SHEAF THEORY

> My mind," he said, "rebels at stagnation. Give me problems, give me work, give me the most abstruse cryptogram or the most intricate analysis, and I am in my own proper atmosphere. I can dispense then with artificial stimulants. But I abhor the dull routine of existence. I crave for mental exaltation. That is why I have chosen my own particular profession, or rather created it, for I am the only one in the world.
>
> (Sherlock Holmes - Arthur Conan Doyle, The Sign of Four)

2.1 *Presheaves, Sheaves and Étale Spaces*

Sheaf theory concerns itself with tracking down locally defined data attached to open subsets of a topological space. In other words, sheaf theory is a tool specifically designed to deal with questions of locality.

Definition 3 (Presheaf)**.** Let X be a topological space and let $\mathbf{t}$ be the category of open subsets of X with inclusion maps as morphisms. Let $\mathscr{C}$ be an arbitrary category and associate to each object $U \in \mathbf{t}$ an object $S(U) \in \mathscr{C}$. Moreover, to each inclusion $\mathbf{t} \ni i_V^U : V \hookrightarrow U$ associate a morphism (called a restriction map) $\mathscr{C} \ni \varrho_V^U := S(i_V^U) : S(U) \to S(V), s \mapsto s \restriction_V$ such that

$$\varrho_U^U = \mathrm{id}_{S(U)} \qquad \varrho_W^V \circ \varrho_V^U = \varrho_W^U$$

whenever $\mathbf{t} \ni W \subset V$. The family $S = \{S(U), \varrho_V^U\}$ is called a presheaf of the category $\mathscr{C}$ on X. Put differently, a presheaf is just a contravariant functor of the category $\mathbf{t}$ into the category $\mathscr{C}$. If we are given yet another presheaf $S' = \{S'(U), (\varrho')_V^U\}$ then a presheaf map is just a natural transformation $\varphi : S' \to S$. In other words, a presheaf map $\varphi : S' \to S$ is a family $\{\varphi_U\} \subset \mathscr{C}$ of morphisms $\varphi_U : S'(U) \to S(U)$ such that $\varphi_V \circ (\varrho')_V^U = \varrho_V^U \circ \varphi_U$ whenever $V \subset U$. Certainly, presheaves on X together with presheaf maps as morphisms form a category.

Definition 4 (Sheaf)**.** A presheaf $S = \{S(U), \varrho_V^U\}$ of the category $\mathscr{C}$ on X is called a sheaf (of the category $\mathscr{C}$ on X) if Serre's condition is satisfied, that is, given an open set $U \in \mathbf{t}$ and an open partition $\{U_\alpha\}$ of U and elements $s_\alpha \in S(U_\alpha)$ such that

$$s_\alpha \restriction_{U_\alpha \cap U_\beta} = s_\beta \restriction_{U_\alpha \cap U_\beta} \tag{19}$$

for all α, β then there exists exactly one element $s \in S(U)$ such that $s \restriction_{U_\alpha} = s_\alpha$ for all α.

In what is to come we will usually consider (pre)sheaves of the category of sets, abelian groups or rings etc. In that spirit we will just refer to these as (pre)sheaves of sets/abelian groups/rings on X.

Definition 5 (Étale Space)**.** Let X be a topological space. A pair $(\mathscr{S}, \zeta)$ is called an Étale space over X if $\mathscr{S}$ is a topological space and $\zeta : \mathscr{S} \to X$ is a local homeomorphism. The fiber $\mathscr{S}_x := \zeta^{-1}\{x\}$ for $x \in X$ is referred to as the stalk of $\mathscr{S}$ at x. Continuous maps

$$s : U \underset{\text{open}}{\subset} X \to \mathscr{S} \qquad s \mapsto s_x$$

with $\zeta \circ s = \mathrm{id}_U$ are called sections in $\mathscr{S}$ over X and the value s_x is called the germ of s at x. Sometimes we will simply write $\mathscr{S}$ instead of $(\mathscr{S}, \zeta)$. An Étale morphism f is a map $f : (\mathscr{S}, \zeta) \to (\mathscr{S}', \zeta')$ such that $\zeta = \zeta' \circ f$. Of course, Étale spaces together with Étale morphisms yield a category. Étale spaces may also be endowed with algebraic structure. For example the Étale space $\mathscr{S}$ is called an Étale space of abelian groups, if for all $x \in X$ the stalks $\mathscr{S}_x$ are abelian groups such that the maps

$$\mathscr{S} \times_X \mathscr{S} := \bigcup_{x \in X} \mathscr{S}_x \times \mathscr{S}_x \to \mathscr{S} \qquad \mathscr{S}_x \ni (s_x, t_x) \mapsto s_x - t_x \tag{20}$$

$$X \to \mathscr{S} \qquad x \mapsto 0_x \tag{21}$$

are continuous (where $0_x \in \mathscr{S}_x$ denotes the neutral element). For such an Étale space of abelian groups, an Étale homomorphism is defined to be an Étale morphism which respects the underlying algebraic structure (the given map is a group homomorphism on every stalk). Analogously, Étale spaces of abelian groups form a category with morphisms being Étale homomorphisms. More generally, an Étale space of the category $\mathscr{C}$ is just an Étale space such that all stalks are objects of $\mathscr{C}$ and if the objects in $\mathscr{C}$ are endowed with some algebraic structure then the respective algebraic operations are assumed to be continuous and Étale homomorphisms are Étale morphisms respecting the given structure.

From the definition of an Étale space $\mathscr{S}$ we can immediately deduce some simple consequences:

- If $U \subset X$ is open, then $\mathscr{S}_U := \zeta^{-1}(U) \subset \mathscr{S}$ gives rise to a sub-Étale space.
- Every element $s \in \mathscr{S}_x$ is in the image of some section: Since ζ is a local homeomorphism there exists an open neighborhood $T \subset \mathscr{S}$ of s such that $\zeta \restriction_T$ is a homeomorphism onto its image. We certainly have $(\zeta \restriction_T)^{-1} \in \mathscr{S}(\zeta(T))$ and clearly s is in the range of this map.
- Every Étale space determines a sheaf of sets: Indeed, let $\mathscr{S}(U)$ denote the set of all sections $s\colon U \to \mathscr{S}$ and define ϱ_V^U to be the restriction map $s \mapsto s \restriction V$. Then, of course, $\{\mathscr{S}(U), \varrho_V^U\}$ is a sheaf of sets and we will refer to this particular sheaf by the sheaf of sections in $\mathscr{S}$.
- The open sets of $\mathscr{S}$, which are projected homeomorphically onto open subsets of X by ζ, form a base for the open sets of $\mathscr{S}$: Let $S \subset \mathscr{S}$ be open and pick $s \in S$. Now by assumption there exists an open set $T \subset \mathscr{S}$ which contains s and such that the restricted map $\zeta \restriction_T$ is a homeomorphism onto its image. But then $s \in S \cap T \subset S$ is open and $\zeta \restriction_{S\cap T}$ also yields a homeomorphism onto its image.
- The subspace topology on $\mathscr{S}_x$ is the discrete topology: By the preceding bullet point a basis for the topology on $\mathscr{S}_x$ is given by open sets T which are homeomorphically projected onto open subsets of X by ζ intersected with $\mathscr{S}_x$. However, in that case the restricted map $\zeta \restriction_{T\cap\mathscr{S}_x}\colon T \cap \mathscr{S}_x \to \{x\}$ is bijective and therefore $T \cap \mathscr{S}_x$ is just a single point.
- Let U be an open neighborhood of $x \in X$ and take $s,t \in \mathscr{S}(U)$ (= sections on U). Then $s_x = t_x$ if and only if there exists a neighborhood V of x such that $s \restriction_V = t \restriction_V$: Since ζ is a local homeomorphism there has to exist an open neighborhood $T \subset \mathscr{S}$ such that $s_x \in T$ and $\zeta \restriction_T$ is a homeomorphism onto its image. We can then consider the open set $\tilde{T} := s^{-1}(T) \cap t^{-1}(T)$. By definition of what it means to be a section we have $\zeta \circ s \restriction_{\tilde{T}} = \mathrm{id}_{\tilde{T}} = \zeta \circ t \restriction_{\tilde{T}}$ and thus since ζ restricted to T is a homeomorphism, we get $s \restriction_{\tilde{T}} = t \restriction_{\tilde{T}}$.
- If $\mathscr{S}$ is an Étale space of abelian groups (or rings etc.), then $f \pm g \in \mathscr{S}(U)$ is well defined for all $f,g \in \mathscr{S}(U)$.
- For two Étale spaces $(\mathscr{S},\zeta)$ and $(\mathscr{S}',\zeta')$ over X the fiber product
$$\mathscr{S} \times_X \mathscr{S}' := \{(s,s') \in \mathscr{S} \times \mathscr{S}' \mid \zeta(s) = \zeta'(s')\} = \bigcup_x (\mathscr{S}_x \times \mathscr{S}'_x) \tag{22}$$
together with the map $\mathscr{S} \times_X \mathscr{S}' \to X, (s,s') \mapsto \zeta(s)$ is an Étale space.

A natural question to ask is whether or not a presheaf induces an Étale space. Answering this question is not all that difficult (the answer is yes as the next proposition will show). However, the follow-up question of whether one can recover the original (pre)sheaf after one has turned it into an Étale space is by far more involved and is quite technical and tedious. Surprisingly enough, it actually turns out that Serre's condition is essential and that one can actually verify that Étale spaces and sheaves are essentially the same (from a categorical point of view).

Proposition 1. *Every presheaf S gives rise to an Étale space $\mathscr{S}$ and a presheaf map $S \to \{\mathscr{S}(U), \varrho_V^U\}$. In particular, the presheaf map $S \to \{\mathscr{S}(U), \varrho_V^U\}$ is an isomorphism if and only if S is a sheaf.*

Proof. Consider the disjoint union $\mathscr{U}_x := \bigsqcup S(U)$ where the union is taken over all open sets $U \subset X$ which contain x. We then define the stalk $\mathscr{S}_x$ at x to be the direct limit $\varinjlim S(U)$. We will unwrap now swiftly what is meant by taking the direct limit. Let U, V be open subsets of X containing x and take $s \in S(U)$ and $t \in S(V)$. We say that s and t are equivalent in $\mathscr{U}_x$ if there exists an open neighborhood $W \subset X$ of x such that $W \subset U \cap V$ and $s \restriction_W = t \restriction_W$ (s and t are eventually the same, loosely speaking).
Claim: This defines an equivalence relation $\sim$ on $\mathscr{U}_x$:
The relation $\sim$ is certainly reflexive and symmetric. So let $s \in S(U), t \in S(V)$ and $u \in S(W)$ such that $s \sim t$ and $t \sim u$. By assumption there exist open neighborhoods $W_1 \subset U \cap V$ and $W_2 \subset V \cap W$ of x such that $s \restriction_{W_1} = t \restriction_{W_1}$ and $t \restriction_{W_2} = u \restriction_{W_2}$. Set $X := W_1 \cap W_2$, then we obtain

$$s \restriction_X = \varrho_X^U\, s = \varrho_X^{W_1} \varrho_{W_1}^U\, s = \varrho_X^{W_1} \varrho_{W_1}^V\, t = \varrho_X^V\, t = \varrho_X^{W_2} \varrho_{W_2}^V\, t = \varrho_X^{W_2} \varrho_{W_2}^W\, u = u \restriction_X$$

and therefore $s \sim u$ which proves the claim.

Now just define

$$\mathscr{S}_x := \mathscr{U}_x / \sim$$

which exactly recovers the meaning of the direct limit from before. The next step is to collect all these stalks to obtain our, still topologically naked, Étale space

$$\mathscr{S} := \bigsqcup_{x \in X} \mathscr{S}_x$$

which of course is endowed with a natural projection $\zeta \colon \mathscr{S} \to X$. This yields natural maps $\varrho_x^U \colon S(U) \to \mathscr{S}_x$ mapping each element $s \in S(U)$ onto its equivalence class $\varrho_x^U\,(s) \in \mathscr{S}_x$ (for $x \in U$). Each $s \in S(U)$ now induces a map $\bar{s} \colon U \to \mathscr{S}$ given by $x \mapsto \varrho_x^U\,(s)$. If $\mathbf{t}$ is again the category of open sets in X, then the family of sets $\{\bar{s}(U) \mid U \in \mathbf{t}, s \in S(U)\}$ gives rise to a topology on $\mathscr{S}$ such that $\zeta \colon \mathscr{S} \to X$ is a local homeomorphism. Therefore $(\mathscr{S}, \zeta)$ is an Étale space. The maps $\bar{s}$ are sections in $\mathscr{S}$and therefore we may define $\varphi_U \colon S(U) \to \mathscr{S}(U)$ by $s \mapsto \bar{s}$. In order to show that $\varphi = \{\varphi_U\}$ defines a sheaf map $S \to \{\mathscr{S}(U), \varrho_V^U\}$ we have to verify that the diagram

$$\begin{array}{ccc} S(U) & \xrightarrow{S(i_V^U)} & S(V) \\ \big\downarrow{\scriptstyle \varphi_U} & & \big\downarrow{\scriptstyle \varphi_V} \\ \mathscr{S}(U) & \xrightarrow[\varrho_V^U]{} & \mathscr{S}(V) \end{array}$$

commutes, where $\mathbf{t} \ni i_V^U \colon V \hookrightarrow U$. However, for arbitrary $s \in S(U)$ and $x \in V$ we obtain

$$\varphi_V \circ S(i_V^U)(s)(x) = \overline{s \restriction_V}(x) = \varrho_x^V\,(s \restriction_V) \overset{s \sim s\restriction_V}{=} \varrho_x^U\,(s) = \varrho_V^U \circ \varphi_U(s)(x) \tag{23}$$

Assume now that S is a sheaf and fix an open set $U \in \mathbf{t}$. We first verify injectivity of φ_U. The equation $\varphi_U(s) = \varphi_U(t)$ for $s, t \in S(U)$ is equivalent to the statement that for all $x \in U$ there exists an open neighborhood $W_x \subset U$ of x such that $s \restriction_{W_x} = t \restriction_{W_x}$. The family $\{W_x\}$ gives rise to an open covering of U and thus by Serre's condition $s = t$. For surjectivity let $f \in \mathscr{S}(U)$ be arbitrary. By construction of $\mathscr{S}$ for every $x \in U$ there exists an open neighborhood $U_x \subset U$ of x and an element $s^x \in S(U_x)$ such that $\varrho_x^{U_x}\, s^x = f(x)$. Now as ζ is a local homeomorphism there exists an open neighborhood $V_x \subset U_x$ of x such that $\zeta \restriction_{\tilde{V}_x}$ is a homeomorphism onto its image, where $\tilde{V}_x := \zeta^{-1}(V_x)$. By setting $s^x := s^x \restriction_{V_x}$ we may assume without loss of generality that $s^x \in S(V_x)$. Since $\zeta(\varrho_y^{V_x}\, s^x) = y$ for all $y \in V_x$ we have $(\zeta \restriction_{\tilde{V}_x})^{-1} = \left(V_x \to \mathscr{S}, y \mapsto \varrho_y^{V_x}\, s^x\right) \in \mathscr{S}(V_x)$. In particular, for all $y \in V_x$ we have

$$f(y) = f(\zeta(\varrho_y^{V_x}\, s^x)) = \varrho_y^{V_x}\, s^x$$

Furthermore note that for $z \in V_x \cap V_y$ we have

$$\varphi_{V_x \cap V_y}(\varrho_{V_x \cap V_y}^{V_x}\, s^x)(z) = \varrho_z^{V_x}\, s^x = f(z) = \varrho_z^{V_y}\, s^y = \varphi_{V_x \cap V_y}(\varrho_{V_x \cap V_y}^{V_y}\, s^y)(z)$$

and therefore by injectivity of $\varphi_{V_x \cap V_y}$ we have $\varrho^{V_x}_{V_x \cap V_y} s^x = \varrho^{V_y}_{V_x \cap V_y} s^y$. Since $\{V_x\}$ is also a cover of U Serre's condition yields a unique element $s \in S(U)$ such that $s \restriction_{V_x} = s^x$ for all $x \in U$. At long last we can then deduce that for all $x \in U$ we have

$$f(x) = \varrho^{V_x}_x s^x = \bar{s}(x)$$

This shows that φ_U is indeed an isomorphism. Conversely, suppose all $\{\varphi_U\}$ are isomorphisms. If $U \in \mathbf{t}$ has an open partition $\{U_\alpha\}$ with $s_\alpha \in S(U_\alpha)$ such that $s_\alpha \restriction_{U_\alpha \cap U_\beta} = s_\beta \restriction_{U_\alpha \cap U_\beta}$ then let $f_\alpha := \varphi_{U_\alpha}(s_\alpha) \in \mathscr{S}(U_\alpha)$. It is easily verified that $f := \bigcup f_\alpha$ yields a well defined map $U \to \mathscr{S}$ and in particular $f \in \mathscr{S}(U)$. As φ_U is an isomorphism we may then define $s := \varphi_U^{-1}(f) \in S(U)$. By using the fact that φ is a natural transformation we infer

$$s \restriction_{U_\alpha} = S(i^U_{U_\alpha})(\varphi_U^{-1}(f)) = \varphi_{U_\alpha}^{-1}(\varrho^U_{U_\alpha}(f)) = \varphi_{U_\alpha}^{-1}(f_\alpha) = s_\alpha$$

showing that Serre's condition holds for S. □

This was, admittedly, extremely tedious. However, the above result is valuable both for the content of its statement and the information contained in the proof, as we will see very soon (I promise). Let us denote by $\Gamma\mathscr{S}$ the sheaf of sections induced by the Étale space $\mathscr{S}$ and denote by $\check{\Gamma}S$ the Étale space induced by the sheaf S. Let us prove another smaller lemma before reaping the fruits of our hard work:

Lemma 4. *For an Étale space $\mathscr{S}$ we have a canonical isomorphism $\mathscr{S} \to \check{\Gamma}\Gamma\mathscr{S}$. Moreover, every Étale morphism $\mathscr{S}' \to \mathscr{S}$ resp. sheaf map $S' \to S$ induces a sheaf map $\Gamma\mathscr{S}' \to \Gamma\mathscr{S}$ resp. Étale morphism $\check{\Gamma}S' \to \check{\Gamma}S$.*

Proof. For the first part of the lemma consider the map $\mathscr{S} \to \Gamma\check{\Gamma}\mathscr{S}$ which takes an element $s \in \mathscr{S}_x$ and maps it onto the equivalence class in $(\Gamma\check{\Gamma}\mathscr{S})_x$ of some section $f \in \check{\Gamma}\mathscr{S}(U) = \mathscr{S}(U)$ with $f(x) = s$. In other words, consider $\mathscr{S} \to \Gamma\check{\Gamma}\mathscr{S}$ given by $s \mapsto \varrho^U_x f$ where $f \in \mathscr{S}(U)$ such that $f(x) = s$. This is well defined, for if $g \in \mathscr{S}(U)$ is another section with $g(x) = s$ we know that $f \restriction_V = g \restriction_V$ for some open neighborhood $V \subset U$ of x. However, this already implies $\varrho^U_x f = \varrho^U_x g$. This map is easily seen to be bijective, which proves the first part of the lemma. Suppose now that $f\colon \mathscr{S}' \to \mathscr{S}$ is an Étale morphism. Define the family $\Gamma(f) = \{\Gamma(f)_U\}$ of maps $\Gamma(f)_U\colon \Gamma\mathscr{S}'(U) \to \Gamma\mathscr{S}(U)$ by $s' \mapsto f \circ s'$. Write $\Gamma\mathscr{S} = \{\Gamma\mathscr{S}(U), \varrho^U_V\}$ and $\Gamma\mathscr{S}' = \{\Gamma\mathscr{S}'(U), \lambda^U_V\}$ for the induced sheaves. Clearly for $s' \in \Gamma\mathscr{S}'(U)$ we have

$$\varrho^U_V \circ \Gamma(f)_U(s') = \varrho^U_V (f \circ s') = f \circ s' \restriction_V = f \circ (s' \restriction_V) = \Gamma(f)_V \circ \lambda^U_V(s') \tag{24}$$

so $\Gamma(f)$ is a sheaf map. Conversely, suppose $\varphi'\colon S' \to S$ is a sheaf map and $S = \{S(U), \varrho^U_V\}$ and $S' = \{S'(U), \lambda^U_V\}$. We will now define a map $\check{\Gamma}(\varphi')\colon \check{\Gamma}(S') \to \check{\Gamma}(S)$. For $s^x \in \check{\Gamma}(S')_x$ there must exist some open neighborhood $U \subset X$ of x along with $s \in S'(U)$ such that $\lambda^U_x s = s^x$. Set $\check{\Gamma}(f)(s^x) := \varrho^U_x \varphi'_U(s)$. This is well defined. Indeed, if $t \in S'(U)$ is another element such that $\lambda^U_x t = s^x$ then by definition of the given equivalence relation there must exist an open neighborhood $W \subset U$ of x such that $\lambda^U_W(s) = \lambda^U_W(t)$. Using the defining properties of a sheaf map one arrives at

$$\varrho^U_W \circ \varphi'_U(s) = \varphi'_W \circ \lambda^U_W(s) = \varphi'_W \circ \lambda^U_W(t) = \varrho^U_W \circ \varphi'_U(t) \tag{25}$$

Thus $\check{\Gamma}(\varphi')$ is well defined and it is then clear by the given construction that $\zeta \circ \check{\Gamma}(\varphi') = \zeta'$ (where ζ resp. ζ' is the projection onto $\check{\Gamma}S$ resp. $\check{\Gamma}S'$). □

We now collect all these facts together. By proposition 1 we have a canonical sheaf isomorphism $S \to \Gamma\check{\Gamma}S$. Analogously, by lemma 4 we have a canonical Étale space isomorphism $\mathscr{S} \to \check{\Gamma}\Gamma\mathscr{S}$. In particular, again by lemma 4 we therefore obtain that Γ and $\check{\Gamma}$ define functors

$$\Gamma\colon \text{Étale spaces} \rightsquigarrow \text{Sheaves} \qquad \check{\Gamma}\colon \text{Sheaves} \rightsquigarrow \text{Étale spaces} \tag{26}$$

Now by what we have seen there are natural isomorphisms of the (functor) compostions $\Gamma\check{\Gamma}$ and $\check{\Gamma}\Gamma$ with the identity. Hence the categories of sheaves of sets and Étale spaces over X are equivalent. Recall also that both (pre)sheaves and Étale spaces may come endowed with some algebraic structure. The construction of proposition 1 applied to a (pre)sheaf with algebraic structure will give rise to an Étale space with the same algebraic structure. Combining all this leads to:

Theorem 4 (Equivalence Principle). *The category of Étale spaces of groups (rings etc.) over X is equivalent to the category of sheaves of groups (rings etc.) over X.*

The preceding theorem tells us that we need not distinguish between an Étale space and its sheaf of sections. Therefore we will sometimes call Étale spaces sheaves and vice versa. Before proceeding even further into sheaf theory it is due time to give some examples of actual sheaves. Throughout, $\mathbb{F}$ could either be the complex number field $\mathbb{C}$ or the real number field $\mathbb{R}$ (unless we explicitly say otherwise).

Examples 1.

1. Let X be a topological space and let $\mathbb{F}$ be a field endowed with the discrete topology. The canonical projection $\zeta\colon \mathbb{F}_X := X \times \mathbb{F} \to X$ turns $\mathbb{F}_X$ into a sheaf of fields. This is referred to as a constant sheaf.

2. Let $X := \mathbb{R}$ and define $S(U) := \mathbb{Z}$ for all open subsets $U \subset X$ and $\varrho_V^U = 0$ for all $V \subset U$ open. Then $S = \{S(U), \varrho_V^U\}$ defines a presheaf on X. However, if $\mathscr{S}$ denotes the induced Étale space then we note that all stalks $\mathscr{S}_x$ consist only of the zero element. Therefore, the sheaf of sections obtained from the Étale space $\mathscr{S}$ satisfies $\mathscr{S}(U) = 0 \neq S(U)$ for all open $U \subset X$.

3. Let X be a topological space and define $\mathscr{C}(U)$ to be the set of continuous functions $X \to \mathbb{F}$ and let ϱ_V^U simply be the restriction maps. Now $\mathscr{C} = \{\mathscr{C}(U), \varrho_V^U\}$ certainly gives rise to a sheaf. The induced Étale space $\mathscr{C}_X$ is called the sheaf of germs of continuous functions over X. The Étale space $\mathscr{C}_X$ is an Étale space of commutative rings. However, note that $\mathscr{C}_X$ is not Hausdorff in general: Take $X = \mathbb{R}$ with the standard topology and let f_0 denote the germ of continuous functions which are identically 0 in a small neighborhood of $0 \in \mathbb{R}$. Now consider the continuous function $g(x) = e^{-1/x}$ whenever $x > 0$ and $g(x) = 0$ for all $x \leq 0$. The function g also defines a germ g_0 at 0 which is different from f_0. Any couple of open neighborhoods $T_f, T_g \subset \mathscr{C}_X$ of f_0, g_0 must have non-empty intersection since $g(x) = 0$ for all $x < 0$.

4. For an open subset X of $\mathbb{R}^d$ let $\mathscr{C}^\infty(U)$ denote the set of infinitely differentiable functions $U \to \mathbb{F}$ and let ϱ_V^U denote the usual restriction maps. This again yields a sheaf and the Étale space of commutative rings $\mathscr{C}_X^\infty$ thus generated is called the sheaf of germs of smooth functions. Just as in the preceding example, by using the same counterexample, we infer that $\mathscr{C}_X^\infty$ is not Hausdorff. The set X need not be an open subset of $\mathbb{R}^d$: We could also let X be an arbitrary smooth manifold.

5. Let M be a smooth (abstract) manifold and let $\xi\colon M \to TM$ be a vector field (where TM is the tanget bundle of M). Now recall that for every smooth function $f \in \mathscr{C}^\infty(M, \mathbb{R})$ we can consider the smooth function $\xi(f)\colon M \to \mathbb{R}$ given by $\xi(f)(x) := \xi(x)(f)$ (recall that $\xi(x) \in T_xM$ is a derivation at x on the space of differentiable functions $\mathscr{C}^\infty(M, \mathbb{R})$). Now for an open set U of M we may consider the (partial) differential equation $\xi(f) = 0$ for $f \in \mathscr{C}^\infty(M, \mathbb{R})$ and we then let $S(U)$ be the vector space of all those smooth functions f on U such that $\xi(f) = 0$. Again we just choose the standard restriction maps ϱ_V^U and thus obtain a sheaf $\{S(U), \varrho_V^U\}$.

6. For a domain $D \subset \mathbb{C}^d$ let $\mathscr{O}(U)$ denote the set of holomorphic functions $U \to \mathbb{C}$ and let ϱ_V^U denote the usual restriction maps. This is again a sheaf and the induced Étale space $\mathscr{O}_D$ is called the sheaf of germs of holomorphic functions over D. Certainly $\mathscr{O}_D$ is a sheaf of commutative rings.

Lemma 5. *For each domain D in $\mathbb{C}^d$ the sheaf of germs of holomorphic functions $\mathscr{O}_D$ is Hausdorff.*

Proof. Let $f_x, g_y \in \mathscr{O}_D$ be two different germs. For $x \neq y$ the statement is clear. For $x = y$ pick two holomorphic functions $f, g \in \mathscr{O}(B)$ with $\varrho_x^B\, f = f_x$ and $\varrho_x^B\, g = g_x$, where $B \subset D$ is some open disk centered at x. Now we recall that f and g both induce sections $\overline{f}, \overline{g} \in \mathscr{O}_D(B)$. If we had $\overline{f}(B) \cap \overline{g}(B) \neq \emptyset$, then there would exist $z \in U$ such that $\varrho_z^B\, f = \overline{f}(z) = \overline{g}(z) = \varrho_z^B\, g$. But this is equivalent to there existing an open neighborhood $W \subset B$ of z such that $f\restriction_W = g\restriction_W$. By the classical Identitätssatz we thus have $f = g$, since D is connected. However, this would also imply $f_x = g_x$, a contradiction. Therefore, $\overline{f}(B) \cap \overline{g}(B) = \emptyset$ and both these sets are open neighborhoods of f_x resp. g_x. □

2.2 $\mathscr{A}$-modules and Image Sheaves

Definition 6 ($\mathscr{A}$-modules)**.** Let $\mathscr{A}$ be a sheaf of rings on some topological space X and suppose $\mathscr{S}$ is a sheaf of abelian groups on X. The sheaf $\mathscr{S}$ is called an $\mathscr{A}$-module or $\mathscr{A}$-sheaf, if each stalk $\mathscr{S}_x$ is an $\mathscr{A}_x$-module such that the map $\mathscr{A} \times_X \mathscr{S} \to \mathscr{S}$, defined on each stalk by $(a_x, s_x) \mapsto a_x s_x$, is continuous. As one might have guessed, $\mathscr{A}$-morphisms are those sheaf maps $\varphi\colon \mathscr{S}' \to \mathscr{S}$ which induce $\mathscr{A}_x$-linear stalk maps $\varphi_x := \varphi\restriction_{\mathscr{S}'_x}\colon \mathscr{S}'_x \to \mathscr{S}_x$ (Note that we have made use of the Equivalence principle 4 in order to translate this back into the sheaf setting). In that sense $\mathscr{A}$-modules form a category. For every $\mathscr{A}$-module $\mathscr{S}$ on X we define the support of the sheaf $\mathscr{S}$ to be the set

$$\operatorname{supp}\mathscr{S} := \overline{\{x \in X \mid \mathscr{S}_x \neq 0\}} \tag{27}$$

As is common for such algebraic constructions, one might ask for $\mathscr{A}$-submodules or ideals and so on. We have a natural notion of $\mathscr{A}$-submodules, sums of $\mathscr{A}$-submodules, finite intersections of $\mathscr{A}$-submodules of $\mathscr{S}$ and so on (all of this is just given stalkwise for each x).Moreover, if we are given an $\mathscr{A}$-submodule $\mathscr{I}$ of $\mathscr{A}$ itself, then we call $\mathscr{I}$ an ideal in $\mathscr{A}$. We call $\mathscr{I}$ that way, because for an $\mathscr{A}$-module $\mathscr{S}$, we may simply define the product $\mathscr{I}\cdot\mathscr{S}$ stalkwise by $\mathscr{I}_x \cdot \mathscr{S}_x$ and this certainly yields an $\mathscr{A}$-submodule of $\mathscr{S}$. In particular, for a submodule $\mathscr{S}'$ of $\mathscr{S}$ one may consider the set

$$\mathscr{S}/\mathscr{S}' := \bigcup \mathscr{S}_x/\mathscr{S}'_x \tag{28}$$

and endow this with the quotient topology. This means that we define the finest possible topology on $\mathscr{S}/\mathscr{S}'$ such that the quotient map $\pi\colon \mathscr{S} \to \mathscr{S}/\mathscr{S}'$ is continuous. The pair $(\mathscr{S}/\mathscr{S}', \zeta_\pi)$, where we defined $\zeta_\pi(s_x + \mathscr{S}'_x) := x$ stalkwise for all $s_x \in \mathscr{S}_x$, is an Étale space and in particular this defines an $\mathscr{A}$-module (this yields the notion of a quotient $\mathscr{A}$-module).

Definition 7. Suppose $\varphi\colon \mathscr{S} \to \mathscr{S}'$ is an $\mathscr{A}$-morphism. We define

$$\ker\varphi := \bigcup_{x\in X} \ker\varphi_x \tag{29}$$

$$\operatorname{im}\varphi := \bigcup_{x\in X} \operatorname{im}\varphi_x \tag{30}$$

to be the kernel respectively image of φ. The cokernel of φ is the quotient sheaf

$$\operatorname{Coker}\varphi := \mathscr{T}/\operatorname{im}\varphi \tag{31}$$

A sequence of $\mathscr{A}$-morphisms $\mathscr{S} \xrightarrow{\varphi} \mathscr{S}' \xrightarrow{\psi} \mathscr{S}''$ is called exact, if $\ker\psi = \operatorname{im}\varphi$. Every $\mathscr{A}$-morphism $\varphi\colon \mathscr{S} \to \mathscr{T}$ induces two exact $\mathscr{A}$-sequences

$$0 \to \ker\varphi \to \mathscr{S} \to \operatorname{im}\varphi \to 0 \qquad 0 \to \operatorname{im}\varphi \to \mathscr{T} \to \operatorname{Coker}\varphi \to 0 \tag{32}$$

The direct sum $\mathscr{S} \bigoplus \mathscr{T} := \mathscr{S} \times_X \mathscr{T}$ of $\mathscr{A}$-modules with stalks $(\mathscr{S} \bigoplus \mathscr{T})_x := \mathscr{S}_x \bigoplus \mathscr{T}_x$ is an $\mathscr{A}$-module. In particular we may define the $\mathscr{A}$-sheaves $\mathscr{S}^p := \bigoplus_{1\le j\le p} \mathscr{S}$. We also have the notion of a tensor product. Indeed, if $\mathscr{S} = \{\mathscr{S}(U), \varrho^U_V\}$ and $\{\mathscr{T}(U), \lambda^U_V\}$ are $\mathscr{A}$-modules, then we may define

$$\mathscr{S} \otimes_{\mathscr{A}} \mathscr{T} = \{\mathscr{S}(U) \otimes_{\mathscr{A}(U)} \mathscr{T}(U), \varrho^U_V \otimes_{\mathscr{A}(V)} \lambda^U_V\} \tag{33}$$

and this of course induces canonical isomorphisms $(\mathscr{S} \otimes_{\mathscr{A}} \mathscr{T})_x \xrightarrow{\simeq} \mathscr{S}_x \otimes_{\mathscr{A}_x} \mathscr{T}_x$.

If we have a continuous map $f\colon X \to Y$ along with a sheaf of abelian groups $\mathscr{S}$ on X, then one might ask if we can use f so as to construct a new sheaf of groups on Y.

Definition 8 (Image sheaf)**.** Let $f\colon X \to Y$ be a continuous map and let $\mathscr{S}$ be a sheaf of abelian groups on X. If $f(X) \cap V = \emptyset$, then put $\mathscr{S}(f^{-1}(V)) := 0$. The family

$$\{\mathscr{S}(f^{-1}(V)), \varrho^V_W\} \qquad W \underset{\text{open}}{\subset} V \underset{\text{open}}{\subset} Y \qquad \varrho^V_W = \text{ canonical restriction} \tag{34}$$

gives rise to a sheaf of abelian groups on Y, which we will denote by $f_*(\mathscr{S})$ and we will refer to it as the f-image sheaf of $\mathscr{S}$. Quite evidently, its support is given by

$$\operatorname{supp} f_*(\mathscr{S}) = \overline{f(\operatorname{supp}\mathscr{S})} \tag{35}$$

By the preceding definition we have seen that from a continuous map $f\colon X \to Y$ we may generate another sheaf of abelian groups on Y denoted by $f_*(\mathscr{S})$ (assuming $\mathscr{S}$ is a sheaf of abelian groups on X). What about morphisms between abelian groups on X? Let $\varphi\colon \mathscr{S} \to \mathscr{S}'$ be a morphisms between two sheaves of abelian groups. Recall that this is a family $\{\varphi_U \mid U \underset{\text{open}}{\subset} X\}$ such that the diagram

$$\begin{array}{ccc} \mathscr{S}(U) & \xrightarrow{S(i_V^U)} & \mathscr{S}(V) \\ {\scriptstyle\varphi_U}\downarrow & & \downarrow{\scriptstyle\varphi_V} \\ \mathscr{S}'(U) & \xrightarrow[S'(i_V^U)]{} & \mathscr{S}'(V) \end{array}$$

commutes for all $V \subset U$ open in X. We now define the family $f_*(\varphi) = \{\varphi_{f^{-1}(V)} \mid V \underset{\text{open}}{\subset} Y\}$. Since φ is a morphism of sheaves, we must also have that $f_*(\varphi)$ is a morphism of sheaves $f_*(\mathscr{S}) \to f_*(\mathscr{S}')$. Recall that the composition of two sheaf morphisms $\mathscr{S} \xrightarrow{\varphi} \mathscr{S}' \xrightarrow{\varphi'} \mathscr{S}''$ is given by

$$\varphi' \circ \varphi = \{\varphi'_U \circ \varphi_U \mid U \underset{\text{open}}{\subset} X\} \tag{36}$$

Thus it immediately follows that $f_*(\varphi' \circ \varphi) = f_*(\varphi') \circ f_*(\varphi)$. In particular, if $1_{\mathscr{S}}$ is the identity morphism on the sheaf $\mathscr{S}$, then $f_*(1_{\mathscr{S}}) = 1_{f_*(\mathscr{S})}$. We summarize all of this more abstractly:

Proposition 2. *Let $f\colon X \to Y$ be a continuous map, then f induces a functor*

$$f_*\colon \{\textit{(Pre-)Sheaves on } X\} \rightsquigarrow \{\textit{(Pre-)Sheaves on } Y\} \tag{37}$$

One may verify without too much effort that:

Lemma 6. *Every exact sequence*

$$0 \longrightarrow \mathscr{S} \xrightarrow{\varphi} \mathscr{S}' \xrightarrow{\psi} \mathscr{S}'' \longrightarrow 0 \tag{38}$$

induces an exact sequence

$$0 \longrightarrow f_*(\mathscr{S}) \xrightarrow{f_*(\varphi)} f_*(\mathscr{S}') \xrightarrow{f_*(\psi)} f_*(\mathscr{S}'') \tag{39}$$

In other words, f_* is left exact.

2.3 $\mathbb{C}$-*ringed Spaces*

Definition 9 (Ringed Spaces). A tuple $(X, \mathscr{A}_X)$ consisting of a topological space X and a sheaf of rings $\mathscr{A} = \mathscr{A}_X$ on X is called a ringed space. We refer to $\mathscr{A}_X$ as the structure sheaf of X. Sometimes we will simply write X instead of $(X, \mathscr{A}_X)$ and we write $|X|$ for the underlying topological space.

For example, the sheaf of continuous functions $\mathscr{C}_X$ yields a ringed space $(X, \mathscr{C}_X)$.

Let $f\colon X \to Y$ be a continuous map. We now try to motivate the at first rather unintuitive notion of a morphism between ringed spaces. In order to do so we consider the ringed spaces $\mathscr{C}_X$ and $\mathscr{C}_Y$. Fix some open subset $V \subset Y$, then f induces a $\mathbb{C}$-algebra lifting homomorphism

$$f_V^\#\colon \mathscr{C}_Y(V) \longrightarrow f_*(\mathscr{C}_X)(V) = \mathscr{C}_X(f^{-1}(V)) \qquad g \mapsto g \circ f \tag{40}$$

where, strictly speaking, f needs to be restricted to $f^{-1}(V)$. We immediately infer that this map commutes with restrictions. This motivates:

Definition 10. A morphism $(f, f^\#)\colon (X, \mathscr{A}_X) \to (Y, \mathscr{A}_Y)$ between ringed spaces consists of a continuous map $f\colon X \to Y$ and a sheaf map $f^\#\colon \mathscr{A}_Y \to f_*(\mathscr{A}_X)$. To spell it out more concretely, $f^\#$ is a family of ring homomorphisms $\{f_V^\# \mid V \text{ open in } Y\}$ such that

$$\begin{array}{ccc} \mathscr{A}_Y(V) & \xrightarrow{f^\#_V} & f_*(\mathscr{A}_X)(V) \\ {\scriptstyle \mathscr{A}_Y(i^V_W)}\downarrow & & \downarrow{\scriptstyle \mathscr{A}_X\left(i^{f^{-1}(V)}_{f^{-1}(W)}\right)} \\ \mathscr{A}_Y(W) & \xrightarrow[f^\#_W]{} & f_*(\mathscr{A}_X)(W) \end{array}$$

commutes for all open subsets $W \subset V \subset Y$, where we denoted by $\mathscr{A}_Y(i^V_W)\colon \mathscr{A}_Y(V) \to \mathscr{A}_Y(W)$ and $\mathscr{A}_X\left(i^{f^{-1}(V)}_{f^{-1}(W)}\right)\colon \mathscr{A}_X(f^{-1}(V)) \to \mathscr{A}_X(f^{-1}(W))$ the restrictions on the sheaves $\mathscr{A}_Y$ resp. $\mathscr{A}_X$.

Now one might ask how to compose two such morphisms $(X,\mathscr{A}_X) \xrightarrow{(f,f^\#)} (Y,\mathscr{A}_Y) \xrightarrow{(g,g^\#)} (Z,\mathscr{A}_Z)$. However, there is only one sensible definition:

$$(g,g^\#)\circ(f,f^\#) := (g\circ f, g_*(f^\#)\circ g^\#)\colon (X,\mathscr{A}_X) \longrightarrow (Z,\mathscr{A}_Z) \tag{41}$$

By means of Proposition 2 one immediately infers that this is an associative, binary operation. The unit element is clearly given by $(1_X, 1_{\mathscr{A}_X})$. Therefore, the collection of ringed spaces forms a category.

Definition 11 (Locally ringed spaces)**.** We call a ringed space $(X,\mathscr{A})$ a locally ringed space, if every stalk $\mathscr{A}_x$ is a local ring (i.e. it has a unique maximal ideal $\mathfrak{M}(\mathscr{A}_x)$).

Definition 12 ($\mathbb{C}$-ringed Spaces)**.** Let $\mathfrak{K} := X \times \mathbb{C}$ be the constant sheaf of fields on X and let $(X,\mathscr{A})$ be a ringed space. If $\mathscr{A}$ is a $\mathfrak{K}$-module, then we call $\mathscr{A}$ a sheaf of $\mathbb{C}$-algebras. Whenever $\mathfrak{K}$ is a submodule of $\mathscr{A}$, then $1_x \in \mathfrak{K}_x = \mathbb{C}\cdot 1_x$ is the unit of $\mathscr{A}_x$. If that is the case and if, moreover, $(X,\mathscr{A})$ is a locally ringed space with unique maximal ideals $\mathfrak{M}(\mathscr{A}_x)$ so that $\mathscr{A}_x = \mathbb{C}\cdot 1_x \bigoplus \mathfrak{M}(\mathscr{A}_x)$ as $\mathbb{C}$-vector spaces, then we call $\mathscr{A}$ a sheaf of local $\mathbb{C}$-algebras. Now certainly in that case $\operatorname{supp}\mathscr{A} = |X|$. Finally, a ringed space $(X,\mathscr{A})$ is called a $\mathbb{C}$-ringed space, if the structure sheaf $\mathscr{A}$ is a sheaf of local $\mathbb{C}$-algebras.

As $\mathbb{C}$-ringed spaces have much more richness in their constitution as ringed spaces, morphisms between $\mathbb{C}$-ringed spaces must respect the added structure. Recall that if S,R are local rings with unique maximal ideals $\mathfrak{M}_S$ and $\mathfrak{M}_R$, then a local homomorphism $\varphi\colon S \to R$ is a homomorphism such that $\varphi(\mathfrak{M}_S) \subset \mathfrak{M}_R$. Recall that by Lemma 4 every sheaf map induces an Étale map and vice versa. A morphism of locally ringed spaces $(f,f^\#)\colon (X,\mathscr{A}_X) \longrightarrow (Y,\mathscr{A}_Y)$ is a morphism of ringed spaces such that for all $x \in X$ the induced ring map $\mathscr{A}_{Y,f(x)} \longrightarrow \mathscr{A}_{X,x}$ is a local homomorphism. Finally, a morphism of $\mathbb{C}$-ringed spaces $(f,f^\#)\colon (X,\mathscr{A}_X) \longrightarrow (Y,\mathscr{A}_Y)$ is a morphism of locally ringed spaces such that $f^\#\colon \mathscr{A}_Y \longrightarrow f_*(\mathscr{A}_X)$ also defines a $\mathbb{C}$-algebra lifting homomorphism, i.e. a family $\{f^\#_V\colon \mathscr{A}_Y(V) \longrightarrow \mathscr{A}_X(f^{-1}(V)) \mid V \text{ open in } Y\}$ of $\mathbb{C}$-algebra homomorphisms. Certainly enough, both the collection of locally ringed spaces and of $\mathbb{C}$-ringed spaces define categories.

Lemma 7. *Let $(X,\mathscr{A}_X),(Y,\mathscr{A}_Y)$ be locally ringed spaces. If $(f,f^\#)\colon (X,\mathscr{A}_X) \longrightarrow (Y,\mathscr{A}_Y)$ is an isomorphism of ringed spaces, then $(f,f^\#)$ is an isomorphism of locally ringed spaces.*

Proof. This follows immediately by recalling the fact: If S,R are local rings, then any isomorphism of rings $S \to R$ is a local ring homomorphism. □

Examples 2. 1. The sheaf of continuous functions $\mathscr{C}_X$ is a $\mathbb{C}$-ringed space: Indeed, it is clear that $(X,\mathscr{C}_X)$ is a ringed space and that $\mathscr{C}_X$ is a sheaf of $\mathbb{C}$-algebras. Now consider the ring epimorphism

$$\mathscr{E}_x\colon \mathscr{C}_{X,x} \longrightarrow \mathbb{F} \qquad f_x \mapsto f(x) \tag{42}$$

where f is a representative of the germ f_x. Note that this is certainly well defined. The kernel of this map consists exactly of those germs f_x with $f(x) = 0$ (for a representative f of f_x). As the above map is an epimorphism, the isomorphism theorem readily yields

$$\mathscr{C}_{X,x}/\ker(\mathscr{E}_x) \simeq \mathbb{F} \tag{43}$$

So the ideal $\ker(\mathscr{E}_x) = \{f_x \in \mathscr{C}_{X,x} \mid f(x) = 0\}$ is maximal. Moreover, this ideal is the unique maximal ideal in $\mathscr{C}_{X,x}$, since every germ f_x such that $f(x) \neq 0$ has a multiplicative inverse.

2. The sheaf of germs of holomorphic functions $\mathscr{O}_D$, domain $D \subset \mathbb{C}^d$, is a $\mathbb{C}$-ringed space.

Lemma 8. *Every morphism $(f, f^\#)\colon (X, \mathscr{A}_X) \to (Y, \mathscr{A}_Y)$ between $\mathbb{C}$-ringed spaces induces stalk maps $f_y^\#\colon \mathscr{A}_{Y,y} \to f_*(\mathscr{A}_X)_y$ which completely determine $f^\#$. These stalk maps are $\mathbb{C}$-algebra homomorphisms.*

Proof. The first statement was shown in Lemma 4. Since all stalks $\mathscr{A}_{Y,y}$, $f_*(\mathscr{A}_X)_y$ are $\mathbb{C}$-algebras and all $f_V^\#\colon \mathscr{A}_Y(V) \to \mathscr{A}_X(f^{-1}(V))$ are $\mathbb{C}$-algebra homomorphisms, the map $f_y^\#$ is a $\mathbb{C}$-algebra homomorphism for all $y \in Y$. □

Examples 3. 1. The pair $(f, f^\#)\colon (X, \mathscr{C}_X) \to (Y, \mathscr{C}_Y)$ is a morphism between $\mathbb{C}$-ringed spaces.

2. Let X, Y be domains in $\mathbb{C}^n, \mathbb{C}^m$. Pick holomorphic functions $f_1, ..., f_m \in \mathscr{O}(X)$ such that $f\colon X \to \mathbb{C}^m, z \mapsto (f_1(z), ..., f_m(z))$ maps X onto Y. The lifting $f_V^\#\colon \mathscr{C}_Y(V) \to \mathscr{C}_X(f^{-1}(X))$ induces a $\mathbb{C}$-algebra homomorphism $f_V^\#\colon \mathscr{O}_Y(V) \to f_*(\mathscr{O}_X)(V)$. Now certainly the map

$$(f, f^\#)\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y) \tag{44}$$

is a morphism of $\mathbb{C}$-ringed spaces.

3. If $(X, \mathscr{A}_X)$ is a $\mathbb{C}$-ringed space and if U is an open subset in X, then $(U, \mathscr{A}_U)$ is a $\mathbb{C}$-ringed space. The inclusion $i\colon U \hookrightarrow X$ induces a $\mathbb{C}$-algebra lifting homomorphism $i^\#\colon \mathscr{A}_X \to i_*(\mathscr{A}_U)$ (this is the identity on U, the zero map outside U). The space $(U, \mathscr{A}_U)$ together with the inclusion morphism $(i, i^\#)$ is called an open $\mathbb{C}$-ringed subspace of $(X, \mathscr{A}_X)$.

Remark 1. *Let $(f, f^\#)\colon (X, \mathscr{A}_X) \to (Y, \mathscr{A}_Y)$ be a morphism between ringed spaces and let $\mathscr{S}$ be an $\mathscr{A}_X$-module. The sheaf $f_*(\mathscr{S})$ is an $f_*(\mathscr{A}_X)$-sheaf and hence, by means of $f^\#\colon \mathscr{A}_Y \to f_*(\mathscr{A}_X)$, an $\mathscr{A}_Y$-module. Moreover, $\mathscr{A}_X$-morphisms $\varphi\colon \mathscr{S} \to \mathscr{T}$ give rise to $\mathscr{A}_Y$-morphisms $f_*(\varphi)\colon f_*(\mathscr{S}) \to f_*(\mathscr{T})$, so that f_* turns out to be a covariant functor of the category of $\mathscr{A}_X$-modules into the category of $\mathscr{A}_Y$-modules.*

2.4 *Complex Model Spaces and Complex Spaces*

We take a quick look at the even more general notions of Complex Spaces, even though we won't really get into too many details here.

Definition 13. Let $D \subset \mathbb{C}^d$ be a domain and take finitely many holomorphic functions $f_1, ..., f_k \in \mathscr{O}(D)$ and form their ideal sheaf $\mathscr{I} := \mathscr{I}_D := \mathscr{O}_D f_1 + ... + \mathscr{O}_D f_k \subset \mathscr{O}_D$. The quotient sheaf $\mathscr{O}_D / \mathscr{I}_D$ is a sheaf of rings on D. We now put

$$X := \operatorname{supp}(\mathscr{O}_D / \mathscr{I}_D) \qquad \mathscr{O}_X := (\mathscr{O}_D / \mathscr{I}_D)|_X \tag{45}$$

Quite evidently X is the set of common zeros of the holomorphic functions $f_1, ..., f_k$, that is, $X = N(f_1, ..., f_k) = \{x \in D \mid f_1(x) = ... = f_k(x) = 0\}$. One can show that $(X, \mathscr{O}_X)$ is a $\mathbb{C}$-ringed space. This $\mathbb{C}$-ringed space is called the complex model space defined by $\mathscr{I}$ (in D). We write $V(f_1, ..., f_k)$ or simply $V(\mathscr{I})$ for this space.

Now the definition of a complex space is as follows:

Definition 14 (Complex Spaces)**.** A $\mathbb{C}$-ringed space $(X, \mathscr{O}_X)$ is called a complex space, if X is a Hausdorff space and if every point of X has an open neighborhood U such that the open $\mathbb{C}$-ringed subspace $(U, \mathscr{O}_U)$ of $(X, \mathscr{O}_X)$ is isomorphic to a complex model space.

Therefore, locally speaking, complex spaces are determined by finitely many holomorphic functions defined in domains of number spaces. In a complex space (X, O_X) every open subset $U \subset X$ defines an open complex subspace $(U, \mathscr{O}_U)$. Complex spaces form a full subcategory of the category of $\mathbb{C}$-ringed spaces. Morphisms (isomorphisms) between such spaces are called holomorphic (biholomorphic) maps; $\mathscr{O}_X$ modules on a complex space $(X, \mathscr{O}_X)$ are called analytic sheaves on X.

3 CALCULUS OF (COHERENT) SHEAVES - SHEAF YOGA

"Do you wish me a good morning, or mean that it is a good morning whether I want it or not; or that you feel good this morning; or that it is a morning to be good on?"

(Gandalf - J.R.R. Tolkien, The Hobbit, or There and Back Again)

3.1 *Finite and relationally finite Sheaves*

As Remmert puts it, Coherence is, in a vague sense, a principle of analytic continuation from a point to a neighborhood. We now try to build up to the definition of Coherence. Whenever we have a space endowed with some algebraic structure, there is almost always the concept of a basis (e.g. for vector spaces). We also want to have something of this sort for $\mathscr{A}$-modules $\mathscr{S}$, where $(X,\mathscr{A})$ is a ringed space (we will always assume $\mathscr{A}$ to be a sheaf of unital rings). If $p \geq 1$ is some integer, then $\mathscr{A}^p$ is also an $\mathscr{A}$-module and therefore $\mathscr{A}^p(X)$ is an $\mathscr{A}(X)$-module. By definition $\mathscr{A}(X)$ is a unital ring, so there exists a unit section $1 \in \mathscr{A}(X)$. The map

$$\mathscr{A}(X) \longrightarrow \mathscr{A}_x \qquad s \mapsto s(x) \tag{46}$$

is a unital ring homomorphism for all $x \in X$ and therefore $1(x) = 1_x \in \mathscr{A}_x$ is the unit element in $\mathscr{A}_x$ for all $x \in X$. Now for $1 \leq i \leq p$ we define the elements

$$e_i := (0,...,1,...,0) \in \mathscr{A}^p(X) \tag{47}$$

where $0 \in \mathscr{A}(X)$ is just the section which maps x onto the zero element $0_x \in \mathscr{A}_x$ for all $x \in X$. From the construction above it is clear that the Étale map

$$\mathscr{A}^p \to \mathscr{A}^p \qquad \mathscr{A}^p_x \ni (a_{1x},...,a_{px}) \mapsto \sum a_{ix}e_{ix} \in \mathscr{A}^p_x \tag{48}$$

is an isomorphism (it is actually just the identity). The elements $e_1,...,e_p \in \mathscr{A}(X)$ are the canonical basis of $\mathscr{A}^p(X)$. An $\mathscr{A}$-map $\psi\colon \mathscr{A}^p \longrightarrow \mathscr{S}$, where $\mathscr{S}$ is an $\mathscr{A}$-module, is completely determined by its p values $s_i := \psi(e_i)$. Conversely, if we are given a sequence of sections $s_1,...,s_p \in \mathscr{S}(X)$, then this defines an $\mathscr{A}$-map

$$\psi\colon \mathscr{A}^p \longrightarrow \mathscr{S} \qquad (a_{1x},...,a_{px}) \mapsto \sum_1^p a_{ix}s_{ix} \tag{49}$$

The sections s_i are said to generate $\mathscr{S}_x$ resp. $\mathscr{S}$, if $\psi(\mathscr{A}^p)_x = \mathscr{S}_X$ resp. $\psi(\mathscr{A}^p) = \mathscr{S}$.

Definition 15 (Finite $\mathscr{A}$-modules)**.** Let $(X,\mathscr{A})$ be a ringed space. An $\mathscr{A}$-module $\mathscr{S}$ is said to be (locally) finite over $\mathscr{A}$ if for every $x \in X$ there exists an open neighborhood U of x such that $\mathscr{S}_U$ is generated by finitely many sections of $\mathscr{S}(U)$.

Examples 4. All the sheaves $\mathscr{A}^p$ for $1 \leq p < \infty$ are finite. All quotient sheaves of finite sheaves are finite.

Proposition 3. *If $\mathscr{S}$ is a finite $\mathscr{A}$-sheaf, then*

$$supp(\mathscr{S}) = \{x \in X \mid \mathscr{S}_x \neq 0\} \tag{50}$$

Proof. Let $x \in X$ be such that $\mathscr{S}_x = 0$. Choose an open neighborhood U of x and sections $s_1,...,s_p \in \mathscr{S}(U)$ which generate $\mathscr{S}_U$. Since $s_i(x) = 0$ for all i, we may make U smaller such that $s_i = 0$ on U for all i. But then $U \subset X \backslash \{y \in X \mid \mathscr{S}_y \neq 0\}$. □

Proposition 4. *Let $\mathscr{S}$ be a $\mathscr{A}$-module. The following is true:*

1. *If $\mathscr{S}$ is finite and $\mathscr{S}' \overset{\varphi}{\to} \mathscr{S} \overset{\psi}{\to} \mathscr{S}''$ is an $\mathscr{A}$-sequence, then $\{x \in X \mid im\varphi_x = \mathscr{S}_x\}$ and $\{x \in X \mid \psi_x = 0\}$ are open in X.*

2. *If $\mathscr{S}$ is finite and $s_1,...,s_p \in \mathscr{S}(V)$ generate the stalk $\mathscr{S}_x$ for some $x \in X$, then x has an open neighborhood $U \subset V$ such that the maps $s_i \restriction_U$ generate $\mathscr{S}_U$.*

3. *Suppose $\mathscr{S}_1, \mathscr{S}_2$ are submodules of $\mathscr{S}$. If $\mathscr{S}_1$ is finite, the set $\{x \in X \mid \mathscr{S}_{1x} \subset \mathscr{S}_{2x}\}$ is open in X. If $\mathscr{S}_2$ is finite too, the set $\{x \in X \mid \mathscr{S}_{1x} = \mathscr{S}_{2x}\}$ is open in X.*

Proof. Applying Proposition 3 to $\mathscr{S}/\mathrm{im}\varphi$ resp. $\mathscr{S}/\ker\psi$ immediately yields the first claim. Now for the second statement define $\varphi: \mathscr{A}^p \to \mathscr{S}$ by $(a_{1x},...,a_{px}) \mapsto \sum_1^p a_{ix}s_i(x)$. By the first statement we have that $\{x \in X \mid \mathrm{im}\varphi_x = \mathscr{S}_x\}$ is open, which is exactly the content of 2). The last statement follows upon noting that $\{x \in X \mid \mathscr{S}_{1x} \subset \mathscr{S}_{2x}\} = X \backslash [\mathrm{supp}\mathscr{S}_1/(\mathscr{S}_1 \cap \mathscr{S}_2)]$. □

Definition 16. An $\mathscr{A}$-sheaf $\mathscr{S}$ is called relationally finite if for every open set U in X and every $\mathscr{A}_U$-homomorphism $\mathscr{A}_U^p \to \mathscr{S}$ the kernel is finite on U. Phrased differently, if for every set $s_1,...,s_p \in \mathscr{S}(U)$ the sheaf of relations

$$\mathscr{R}(s_1,...,s_p) := \bigcup_{x \in U} \left\{ (a_{1x},...,a_{px}) \in \mathscr{A}_x^p : \sum_1^p a_{ix}s_i(x) = 0 \right\} \tag{51}$$

is finite on U.

It is evident from the definition that subsheaves of relationally finite sheaves are relationally finite. However, it is not true in general that quotient sheaves of relationally finite sheaves are relationally finite.

Lemma 9. *If $\mathscr{S}'$ is a finite submodule of a relationally finite module $\mathscr{S}$, then the quotient module $\mathscr{S}/\mathscr{S}'$ is relationally finite.*

Proof. Fix some open subset U of X and let $\tilde{s}_1,...,\tilde{s}_p \in \mathscr{S}/\mathscr{S}'(U)$. We have to show that the sheaf of relations

$$\mathscr{R}^{\mathscr{S}/\mathscr{S}'}(\tilde{s}_1,...,\tilde{s}_p) = \bigcup_{x \in U} \left\{ (a_{1x},...,a_{px}) \in \mathscr{A}_x^p : \sum_1^p a_{ix}\tilde{s}_i(x) = 0 \right\} \tag{52}$$

is finite on U. Let $x \in U$. For every $1 \le i \le p$ we have $\tilde{s}_i(x) \in \mathscr{S}_x/\mathscr{S}'_x$ and therefore we may write $\tilde{s}_i(x) = s_i(x) + \mathscr{S}'_x$ where s_i is some section on $\mathscr{S}$. If $\pi: \mathscr{S} \to \mathscr{S}/\mathscr{S}'$ is the quotient map, then we must also have that the map $\pi \circ s_i$ is a section on $\mathscr{S}/\mathscr{S}'$ which agrees with $\tilde{s}_i$ in the point x. Thus there exists an open neighborhood $V \subset U$ of x such that $\pi \circ s_i = \tilde{s}_i$ on V. Now since $\mathscr{S}$ is relationally finite we know that

$$\mathscr{R}^{\mathscr{S}}(s_1,...,s_p) = \bigcup_{y \in V} \left\{ (a_{1y},...,a_{py}) \in \mathscr{A}_x^p : \sum_1^p a_{iy}s_i(x) = 0 \right\} \tag{53}$$

is finite. Thus there exists an open neighborhood $W \subset V$ of x and sections $t_1,...,t_l \in \mathscr{R}^{\mathscr{S}}(s_1,...,s_p)(W)$ which generate $\mathscr{R}^{\mathscr{S}}(s_1,...,s_p)_W$. Using finiteness of $\mathscr{S}'$ we may assume without loss of generality that there are sections $s'_1,...,s'_k \in \mathscr{S}'(W)$ which generate $\mathscr{S}'_W$. Note that the map

$$\psi': \mathscr{A}_W^p \longrightarrow \mathscr{S}'_W \qquad (a_{1y},...,a_{ky}) \mapsto \sum_1^k a_{iy}s'_i(y) \tag{54}$$

is an isomorphism of sheaves. Taking an element $(a_{1y},...,a_{py}) \in \mathscr{R}^{\mathscr{S}/\mathscr{S}'}(\tilde{s}_1,...,\tilde{s}_p)$ we observe that

$$0 = \sum a_{iy}\tilde{s}_i(y) = \sum a_{iy}s_i(y) + \mathscr{S}'_y \iff \sum a_{iy}s_i(y) \in \mathscr{S}'_y \tag{55}$$

As $\mathscr{S}$ is relationally finite, all tuples $(a_{1y},...,a_{py}) \in \mathscr{A}_y^p$, which satisfy $\sum a_{iy}s_i(y) = 0$, are finitely generated. Moreover, by finiteness of $\mathscr{S}'$ all tuples $(a_{1y},...,a_{py}) \in \mathscr{A}_y^p$ such that $\sum_1^p a_{iy}s_i(y) \in \mathscr{S}_y \backslash 0$ are finitely generated (this follows from the isomorphism ψ'). Thus we conclude that $\mathscr{S}/\mathscr{S}'$ is relationally finite. □

3.2 *Coherent Sheaves*

Definition 17. An $\mathscr{A}$-module $\mathscr{S}$ is called coherent, or more precisely, $\mathscr{A}$-coherent, if $\mathscr{S}$ is finite and relationally finite. The property of coherence is local in character, thus it makes sense to call a sheaf coherent at $x \in X$ if $\mathscr{S}_U$ is coherent for an open neighborhood U of x.

Proposition 5. *Suppose $\mathscr{S}$ and $\mathscr{T}$ are coherent $\mathscr{A}$-sheaves. Then every $\mathscr{A}_x$-map $\varphi_x\colon \mathscr{S}_x \to \mathscr{T}_x$ extends into a neighborhood U of x to an $\mathscr{A}_U$-map $\varphi\colon \mathscr{S}_U \to \mathscr{T}_U$.*

Proof. For small enough U there are morphisms $\psi\colon \mathscr{A}_U^p \to \mathscr{S}_U$ and $\chi\colon \mathscr{A}_U^p \to \mathscr{T}_U$ such that $\psi(\mathscr{A}_U^p) = \mathscr{S}_U$ and $\chi_x = \varphi_x \circ \psi_x$ (by finiteness of $\mathscr{S}$ and $\mathscr{T}$). By construction we then certainly have $\ker\psi_x \subset \ker\chi_x$. By item 3) of Proposition 4 we have, since the relation sheaf $\mathscr{R}(\psi(e_1),...,\psi(e_p)) = \ker\psi$ (where the e_i are the canonical basis for $\mathscr{A}^p$) is finite by assumption, that $\ker\psi \subset \ker\chi$ for a sufficiently small U. Now we identify $\mathscr{S}_U = \mathscr{A}_U^p/\ker\psi$ and notice that the unique $\mathscr{A}_U$-map $\varphi\colon \mathscr{S}_U \to \mathscr{T}_U$ such that

$$\begin{array}{ccc}
\mathscr{A}_U^p & \xrightarrow{\ \pi\ } & \mathscr{A}_U^p/\ker\psi = \mathscr{S}_U \\
 & {\scriptstyle\chi}\searrow \quad \swarrow{\scriptstyle\varphi} & \\
 & \mathscr{T}_U &
\end{array}$$

commutes, agrees with φ_x on the stalk $\mathscr{S}_x$. □

Remark 2. *It follows straightaway that every finite subsheaf of a coherent sheaf is coherent. In particular, if $\mathscr{S}', \mathscr{S}''$ are coherent subsheaves of a coherent sheaf $\mathscr{S}$ and if $\mathscr{I}$ is a finite ideal, then the sheaves $\mathscr{S}' + \mathscr{S}''$ and $\mathscr{I} \cdot \mathscr{S}$ are coherent.*

A most useful theorem is the following:

Theorem 5 (Three Lemma - Serre's Theorem). *Let $0 \to \mathscr{S}' \to \mathscr{S} \to \mathscr{S}'' \to 0$ be an exact sequence of $\mathscr{A}$-sheaves. Then $\mathscr{S}', \mathscr{S}, \mathscr{S}''$ are all coherent if any two of them are coherent.*

Proof. Noguchi page 70, Theorem 3.3.1 □

There are a bunch of immediate corollaries of this Theorem:

Corollary 5. *The direct sum of finitely many coherent sheaves is coherent.*

Proof. The canonical $\mathscr{A}$-sequence $0 \longrightarrow \mathscr{S} \longrightarrow \mathscr{S} \oplus \mathscr{T} \longrightarrow \mathscr{T} \longrightarrow 0$ is exact. □

Corollary 6. *Let $\varphi\colon \mathscr{S} \to \mathscr{T}$ be an $\mathscr{A}$-homomorphism between coherent sheaves. Then the sheaves $\ker\varphi$, $\mathrm{im}\varphi$ and $\mathrm{Coker}\varphi$ are all coherent.*

Corollary 7. *Let $\mathscr{S}' \xrightarrow{\varphi} \mathscr{S} \xrightarrow{\psi} \mathscr{S}''$ be a sequence of coherent sheaves. Then the set of points $x \in X$ such that $\mathscr{S}'_x \to \mathscr{S}_x \to \mathscr{S}''_x$ is exact is open in X. In particular, coherent sheaves are locally isomorphic if they are stalkwise isomorphic.*

Proof. Since $\ker\psi$ and $\mathrm{im}\varphi$ are coherent by the preceding corollary, the set $\{x \in X \mid \mathrm{im}\varphi_x = \ker\psi_x\}$ is open in X by item 3) in Proposition 4. The last claim follows by an application of Proposition 5. □

We have already seen that every coherent $\mathscr{A}$-sheaf is, locally, the cokernel of an $\mathscr{A}$-morphism $\mathscr{A}^q \to \mathscr{A}^p$. By the Three Lemma we also have that the converse holds:

Corollary 8. *Let $\mathscr{A}$ be coherent. Then an $\mathscr{A}$-sheaf $\mathscr{S}$ is coherent, if locally there exists an exact $\mathscr{A}$-sequence $\mathscr{A}^q \to \mathscr{A}^p \to \mathscr{S} \to 0$.*

Corollary 9. *Let $\mathscr{A}$ be coherent and let $\mathscr{I}$ be a finite ideal. Then an $\mathscr{A}/\mathscr{I}$-sheaf $\mathscr{S}$ on X is $\mathscr{A}/\mathscr{I}$-coherent if and only if $\mathscr{S}$ is $\mathscr{A}$-coherent. In particular, $\mathscr{A}/\mathscr{I}$ is a coherent sheaf of rings.*

3.3 *Extension Principle*

Let Y denote a closed subspace of X and let $i: Y \hookrightarrow X$ be the inclusion. For every sheaf $\mathscr{T}$ of groups in Y the image sheaf $i_*\mathscr{T}$ is a sheaf of groups on X characterized by $i_*\mathscr{T}|_Y = \mathscr{T}$ and $i_*\mathscr{T}|_{X\setminus Y} = 0$. The sheaf $i_*\mathscr{T}$ is called the trivial extension of $\mathscr{T}$ to X. If $\mathscr{B}$ is a sheaf of rings on Y and $\mathscr{T}$ is a $\mathscr{B}$-module, then $i_*\mathscr{B}$ is a sheaf of rings on X and $i_*\mathscr{T}$ is a $i_*\mathscr{B}$-module. It is now straightforward to see that the following is valid:

Theorem 6 (Extension Principle). *A $\mathscr{B}$-sheaf $\mathscr{T}$ on Y is $\mathscr{B}$-coherent if and only if $i_*\mathscr{T}$ is $i_*\mathscr{B}$-coherent on X.*

This is the simplest possible form of this principle of extension. If one wants to develop the theory more generally, a refinement of this theorem is needed for $\mathbb{C}$-ringed spaces $(X, \mathscr{A}_X)$. Every ideal $\mathscr{I} \subset \mathscr{A}_X$ gvies rise to the $\mathbb{C}$-ringed space $(Y, \mathscr{A}_Y)$ where $Y := N(\mathscr{I})$ and $\mathscr{A}_Y := (\mathscr{A}_X / \mathscr{I})|_Y$. Certainly $\mathscr{A}_X / \mathscr{I}$ is the trivial extension of $\mathscr{A}_Y$. Thus the trivial extension $i_*\mathscr{T}$ of every $\mathscr{A}_Y$-module $\mathscr{T}$ is an $\mathscr{A}_X / \mathscr{I}$-module. Applying Theorem 6 and Corollary 9 yields:

Theorem 7. *Let $\mathscr{A}_X$ be coherent and suppose $\mathscr{I} \subset \mathscr{A}_X$ is a finite ideal. Then an $\mathscr{A}_Y$-module $\mathscr{T}$ is $\mathscr{A}_Y$-coherent if and only if the trivial extension $i_*\mathscr{T}$ is $\mathscr{A}_X$-coherent.*

Finally the most general form of the extension principle concerns Coherent Analytic Sheaves:

Theorem 8 (Extension Principle for Coherent Analytic Sheaves). *Let $(Y, \mathscr{O}_Y)$ be a closed complex subspace of a complex space $(X, \mathscr{O}_X)$. Then an analytic sheaf $\mathscr{T}$ on Y is $\mathscr{O}_Y$-coherent if and only if the trivial extension $i_*\mathscr{T}$ of $\mathscr{T}$ to X is $\mathscr{O}_X$-coherent.*

4 OKA'S FIRST COHERENCE THEOREM

We start this section by stating the most general version of Oka's first Coherence Theorem:

Theorem 9 (Theorem of Oka). *The structure sheaf $\mathscr{O}_X$ of every complex space X is coherent.*

One may infer from the general version of the Extension Principle that Oka's Theorem follows if we only showed that $\mathscr{O}_{\mathbb{C}^d}$ is coherent. In order to prove that $\mathscr{O}_{\mathbb{C}^d}$ is coherent we need enough cannon fodder:

Definition 18 (Weierstrass Projections). Consider the monic polynomial

$$\omega(z,w) := w^b + a_1(z)w^{b-1} + \ldots + a_b(z) \in \mathscr{O}(D)[w] \tag{56}$$

where $D \subset \mathbb{C}^d$ is a domain and $1 \leq b < \infty$. The attached space $(W, \mathscr{O}_W)$ in $D \times \mathbb{C}$ is called a Weierstrass model space. The projection $D \times \mathbb{C} \to D$ induces the Weierstrass projection $\psi: (W, \mathscr{O}_W) \to (D, \mathscr{O}_D)$.

Every Weierstrass projection $\psi: W \to D$ induces an $\mathscr{O}_D$-homomorphism $\hat{\psi}: \mathscr{O}_D^b \to \psi_*(\mathscr{O}_W)$: For $U \subset D$, and $s = (s_0, \ldots, s_{b-1}) \in \mathscr{O}_D^b(U)$, the polynomial $\sum s_\beta w^{\beta-1} \in \mathscr{O}_D^b(U)[w]$ induces a section in $(\mathscr{O}/\omega\mathscr{O})(U \times \mathbb{C})$ and therefore, by restriction to W, a section $\hat{s} \in (\mathscr{O}/\omega\mathscr{O})(U \times \mathbb{C})|W = \mathscr{O}_W(\psi^{-1}(U)) = \psi_*(\mathscr{O}_W)(U)$. The map $\mathscr{O}_D^b(U) \to \psi_*(\mathscr{O}_W)(U), s \mapsto \hat{s}$ is an $\mathscr{O}_D(U)$-module homomorphism. As these maps are compatible with restrictions they give an $\mathscr{O}_D$-homomorphism $\hat{\psi}: \mathscr{O}_D^b \to \psi_*(\mathscr{O}_W)$. One can actually show that this homomorphism is an $\mathscr{O}_D$-module isomorphism. The crucial ingredient now is:

Lemma 10 (Coherence Lemma). *Let $\psi: (W, \mathscr{O}_W) \to (D, \mathscr{O}_D)$ be a Weierstrass projection, and assume that the sheaf $\mathscr{O}_D$ is coherent. Then the sheaf $\mathscr{O}_W$ is coherent too, and for every coherent $\mathscr{O}_W$-sheaf $\mathscr{S}$ the image sheaf $\psi_*(\mathscr{S})$ is $\mathscr{O}_D$-coherent.*

We will also need:

Lemma 11 (Formal Criterion for Coherence). *Let $\mathscr{A}$ be a Hausdorff sheaf of rings on a topological space X such that all stalks $\mathscr{A}_x$ are integral domains. Then $\mathscr{A}$ is coherent if the following condition is fulfilled:*

- *For any open subset $U \subset X$ and any section $s \in \mathscr{A}(U)$, the sheaf of rings $\mathscr{A}_U / s\mathscr{A}_U$ on U is coherent at every point $x \in U$ where $s_x \neq 0$.*

Theorem 10 (Oka). *$\mathscr{O}_{\mathbb{C}^d}$ is coherent.*

Proof. We prove the statement by induction. The case $d = 0$ is clear. We only need to verify the formal criterion for coherence now. Let $U \subset \mathbb{C}^d$ be open and let $s \in \mathscr{O}(U)$ and $x \in U$ such that the germ at x induced by s is not 0, i.e. $s_x \neq 0$. Without loss of generality we may assume that $x = 0$ and $s(x) = 0$. We can choose coordinates $(z,w) \in \mathbb{C}^{d-1} \times \mathbb{C}$ such that $s(0,w) \neq 0$. By the Preparation Theorem there is a neighborhood D of $0 \in \mathbb{C}^{d-1}$ and a Weierstrass polynomial $\omega = \omega(z,w) \in \mathscr{O}(D)[w]$ such that $s_x\mathscr{O}_x = \omega_x\mathscr{O}_x$. Now we consider the Weierstrass model space $(W, \mathscr{O}_W)$ induced by the Weierstrass polynomial ω and its Weierstrass projection $(W, \mathscr{O}_W) \to (D, \mathscr{O}_D)$. Since $\mathscr{O}_D$ is coherent by induction hypothesis, the sheaf $\mathscr{O}_W$ is coherent by the Coherence Lemma 10. Now by the Extension principle 8 its trivial extension $i_*\mathscr{O}_W = \mathscr{O}_{D\times\mathbb{C}} / \omega\mathscr{O}_{D\times\mathbb{C}}$ is a coherent sheaf of rings. Since $\mathscr{O}_{D\times\mathbb{C}} / \omega\mathscr{O}_{D\times\mathbb{C}}$ and $\mathscr{O}_U / s\mathscr{O}_U$ coincide around x, the sheaf $\mathscr{O}_U / s\mathscr{O}_U$ is indeed coherent. □

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